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Real and Imaginary

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At each moment of time, a single context becomes real. Therefore, every moment divides the class of all contexts into two parts, the ones realized until that moment and the contexts that are not yet realized. The first part represents the past for a certain moment. A proposition is necessarily true at a given moment if it is a consequence of the past. This definition of necessity entails that the modal values of propositions depend on time, namely, the modal value of a proposition can be different for two given moments. For instance, there are propositions so that they are possibly true at one moment and become impossibly true later.

I. Introduction

By the term "context" we mean the set of conditions which determine the truth value of the propositions. Relatively to a context, *x*, a proposition has a unique value of truth. If "xp" means "The proposition *p* is true relatively to the context *x*," then:

$\neg(xp \equiv x\neg p)$, respectively, (1)

$xp \mid\!- \neg x\neg p$ (the principle of contradiction: a proposition and its negation cannot be both true relatively to the same context),

$\neg xp \mid\!- x\neg p$ (the principle of excluded middle: the proposition or its negation is true relatively to the same context).

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The *domain of the proposition p*, *Dp*, includes all the contexts where *p* is true, namely, Dp = {x/xp}. A proposition becomes true when a context belonging to its domain is *real*, when it *takes place*.[2] For each moment, one and only one context is real. It follows that there is a function from time (**T**) to the set of contexts (X)[3], the *reality function*:

$$r: \mathbf{T} \rightarrow X; \qquad (2)$$
$$x = r(t),\ x \text{ is real at the moment } t.$$

At every moment, t_0, the reality function splits up the class of contexts into two regions, the already realized contexts[4], R_0, and the region of the contexts that still are not realized, $\neg R_0$. All the contexts belonging to R_0 represent the past at the moment t_0[5]:

$$R_0x \equiv (Et)((t<t_0) \;\&\; (x = r(t))) \qquad (3)$$

As we have already seen , the truth values of propositions are relative to a context, *xp*. Therefore, the value of a proposition is determined through the *quantifying* the variable *x*, or through its *interpretation*.[6] In the first case, we obtain the *logical value* of a proposition:

1) A *tautology* is true relatively to any context: (*p* is a tautology) ≡ (x)(xp). If *p* is a tautology, then Dp = X and reciprocally.
2) A *contradiction* is false relatively to any context: (*p* is a contradiction) ≡ (x)(x¬p). In this situation, Dp = ∅.
3) A proposition is *factual* if it is true in some contexts and false in other contexts: (*p* is factual) ≡ ((Ex)(xp) & (Ex)(x¬p)).[7] The domain of a factual proposition is neither universal nor void.

If the variable *x* is interpreted over the set of contexts, we reach the truth value of propositions:

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[2] P. V. Inwagen, *Ontology, Identity, and Modality* (Cambridge: Cambridge University Press, 2001), 37.
[3] S. Kaufmann, C. Condoravdi, V. Harizanov, "Formal Approaches to Modality," in *The Expression of Modality*, ed. W. Frawley (Berlin: Walter de Gruyter, 2006), 92.
[4] J. E. Brenner, *Logic in Reality* (Heidelberg: Springer, 2008), 6.
[5] W. G. Lycan, *Modality and Meaning* (Dordrecht: Kluwer Academic Publ., 1994), 12.
[6] S. Popkorn, *First Steps in Modal Logic* (Cambridge: Cambridge University Press, 1994), 15.
[7] Idem, 77.

1) A proposition is *true* at a given time if and only if the actual context at that time belongs to the domain of *p*: (*p* is true at the moment *t*) ≡ ($r(t) \in Dp$).
2) A proposition is false at the moment *t* if and only if it is not true at that moment: (*p* is false at the moment *t*) ≡ $\neg(r(t) \in Dp)$.

The *truth domain* of a proposition *p* at the moment t_0 contains all the realized contexts up that moment belonging to the domain of *p*, namely, $A_0p = (Dp \cap R_0)$, and the *falsity domain* of *p* at the moment t_0, F_0p, is the complementary to the set A_0p:

$$F_0p = \neg A_0p \quad (4)$$
$$F_0p = \neg(Dp \cap R_0)$$
$$F_0p = (Dp \cap \neg R_0) \cup (\neg Dp \cap R_0) \cup (\neg Dp \cap \neg R_0)$$
$$F_0p = (\neg Dp \cup \neg R_0)$$

We notice that the falsity domain of a proposition includes the realized contexts where the proposition is false and all contexts that are not yet realized. Indeed, we cannot say that a proposition would be true at some point if the contexts from its domain had not yet been realized.[8] We can only *imagine* what would happen if such a context were real but, from an imaginary situation, we cannot conclude that the proposition is true.

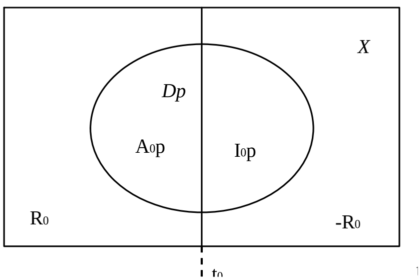

Fig. 1. The domain of a proposition

[8] S. Kaufmann, C. Condoravdi, V. Harizanov, "Formal Approaches to Modality," 93.

For instance, although Romania currently has a president, we can imagine future contexts when it will be ruled by a king, when it will be a kingdom, not a republic. Although the context where Romania is a kingdom could be realized (it even was realized in the past), since such contexts are not taking place now, we cannot say that the proposition "Romania is ruled by a king" is true at the present time, despite the fact that, as we have already noticed, we can imagine such situations. Consequently, besides the truth and falsity domains, we have to recognize the *imaginary domain* of a proposition, containing the contexts where the proposition would be true but they are not real until the moment t_0: $I_0p = Dp \cap \neg R_0$ or $I_0p = F_0p \cap Dp$, (the imaginary domain represents the part of the proposition domain which is still unrealized at a certain moment of time). In this fashion, an imaginary context belonging to the falsity domain of a proposition at a given moment can be included later to the truth domain of that proposition. The relation $I_0p \cup A_0p = Dp$ takes place too, namely, the domain of a factual proposition includes both a real and an imaginary components.

In order to obtain the truth domain of a composed proposition, we have to calculate first its domain and then we will take into account the real part of it:

1) The truth domain of the negation of a proposition is $A_0\neg p = (R_0 \cap D\neg p)$. We notice that the truth domain of negation is not identical with the falsity domain of the proposition but the truth domain of negation is included in it, $A_0\neg p \subset F_0p$. The difference between them coincides to the class of unrealized contexts at that moment:

$$(F_0p - A_0\neg p) \equiv ((\neg R_0 \cup D\neg p) \cap \neg(R_0 \cap D\neg p)) \equiv ((\neg R_0 \cup D\neg p) \cap (\neg R_0 \cup Dp)) \equiv \neg R_0 \quad (5)$$

2) The truth domain of disjunction is: $A_0(p \vee q) = A_0p \cup A_0q = R_0 \cap (Dp \cup Dq)$;

3) The truth domain of conjunction is: $A_0(p \,\&\, q) = A_0p \cap A_0q = R_0 \cap (Dp \cap Dq)$

4) The truth domain of material implication is: $A_0(p \supset q) = F_0p \cap A_0q = R_0 \cap (D\neg p \cup Dq)$ etc.

The truth domain of a tautology at any moment is the whole reality at that moment, respectively, the entire past. If *w* is a tautology, then:

$$Dw = X; \quad (6)$$
$$A_0 w = (R_0 \cap X) = R_0.$$

The imaginary domain of a tautology contains any unrealized context: $I_0 w = \neg R_0 \cap X = \neg R_0$ and it is the same with its falsity domain. Both imaginary and truth domains of a contradiction are void and its falsity domain is identical to *X*, it contains every real or unreal context. Therefore, the contradictions *are not thinkable*; we cannot imagine any situation when a contradiction would take place.

Let us calculate the domains of the expressions "p ∨ ¬p" and "p & ¬p" at a given moment *t*:

$$A_t(p \vee \neg p) = R_t \cap (Dp \cup D\neg p) = R_t \cap X = R_t \quad (7)$$
$$F_t(p \vee \neg p) = \neg R_t \cup (Dp \cap D\neg p) = \neg R_t \cup \varnothing = \neg R_t$$
$$I_t(p \vee \neg p) = \neg R_t \cap (Dp \cup D\neg p) = \neg R_t \cap X = \neg R_t$$
$$A_t(p \,\&\, \neg p) = R_t \cap (Dp \cap D\neg p) = R_t \cap \varnothing = \varnothing$$
$$F_t(p \,\&\, \neg p) = \neg R_t \cup (Dp \cup D\neg p) = \neg R_t \cup X = X$$
$$I_t(p \,\&\, \neg p) = \neg R_t \cap (Dp \cap D\neg p) = \neg R_t \cap \varnothing = \varnothing$$ (a contradiction cannot be imagined in any context).

II. Modalities

A proposition is necessarily true at a given moment, t_0, (it cannot be false at that moment), if and only if it is the consequence of a context realized before t_0. Any consequence of a realized fact shall be realized, namely, it is necessary, and a fact that is not a consequence of a realized fact could realize or not, so that it is not necessary.[9] It follows that the proposition *p* is necessarily true at the moment t_0 if and only if there was a context realized before t_0 so that *p* is a consequence of that context. If "$N_0 p$" means "*p* is necessarily true at the moment t_0", then:

[9] Stalnaker Robert, "The Interaction of Modality with Quantification and Identity", *Modality, Morality, and Belief*, ed. Sinnott Walter-Armstrong, Raffman Diana, Asher Nicholas (Cambridge University Press, Cambridge, 1995), 12.

$N_0p \equiv ((r(t_1) \mid\!- p) \vee (r(t_2) \mid\!- p) \vee \ldots \vee (r(t_n) \mid\!- p))$, where $t_i < t_0$, (8)

$N_0p \equiv ((r(t_1) \,\&\, r(t_2) \,\&\, \ldots \,\&\, r(t_n)) \mid\!- p)$, where $t_i < t_0$,

$(r(t_1) \,\&\, r(t_2) \,\&\, \ldots \,\&\, r(t_n)) =_{not} T_0$, ($T_0$ = the past at the moment t_0),

$N_0p \equiv (T_0 \mid\!- p)$

Having into sight that $\{r(t_1), r(t_2), \ldots, r(t_n)\} = R_0$, it follows the past is just the conjunction of all realized contexts until a certain moment. The past can be imagined correctly only identical to itself. We reach the result that a proposition is necessarily true at a given moment if and only if it is a consequence of the past (or of the reality) at that moment. From the condition of necessity, using the relations between the modal values, it comes the conditions for a proposition to be possible, contingent, or impossible[10]:

1) The proposition *p* is *possibly true* at the moment t_0, M_0p, if it is compatible with the past[11]:

$$M_0p \equiv \neg N \neg p \qquad (9)$$
$$M_0p \equiv \neg(T_0 \mid\!- \neg p) \equiv (T_0 \circ p)$$[12]

2) A proposition is *contingently true* at the moment t_0 if its negation is compatible to the past at t_0:

$$\neg N_0p \equiv \neg(T_0 \mid\!- p) \equiv (T_0 \circ \neg p) \qquad (10)$$

3) The proposition *p* is *impossibly true* at the moment t_0 if its negation is a consequence of T_0:

$$\neg M_0p \equiv N \neg p \qquad (11)$$
$$\neg M_0p \equiv (T_0 \mid\!- \neg p)$$

The modal value of a proposition depends on time.[13] If a proposition, at a given moment, is contingently true, it may become necessarily true

[10] J. C. Beall, B. C. V. Fraassen, *Possibilities and Paradox* (Oxford: Oxford University Press, 2003), 53.
[11] S. Kaufmann, C. Condoravdi, V. Harizanov, "Formal Approaches to Modality," 71.
[12] C. I. Lewis, C. H. Langford, *Symbolic Logic* (Toronto: Dover Publ. Inc., 1959), 153.
[13] A. N. Prior, *Time and Modality* (Oxford: Oxford University Press, 2003), 12.

at a further moment. The following table displays the modal value at the present time for a set of propositions about the color of ravens:

Table 1. Initial modal values

Proposition	Modal value
(A) All ravens are black.	*possibly true*
(E) No raven is black.	*impossibly true*
(I) Some ravens are black.	*necessarily true*
(O) Some ravens are not black.	*contingently true*

The proposition *I* is necessarily true because there are black ravens and the proposition *E* is impossible because it is the negation of *I*. At its turn, *A* is possibly true because it is compatible to the past experience and *O* is contingently true as the negation of *A*.

Let us suppose that, at a given moment in the future, a raven will be of another color than black; it will be blue, for instance. In such a situation, the modal value of the propositions from previous table will be changed as it follows:

Table 2. Final modal values

Proposition	Modal value
(A) All ravens are black.	*impossibly true*
(E) No raven is black.	*impossibly true*
(I) Some ravens are black.	*necessarily true*
(O) Some ravens are not black.	*necessarily true*

We notice that the proposition *A* becomes impossible and the proposition *O* will be necessary. On the other hand, at that further moment, all the propositions from the table (2) will be either necessarily or impossibly true; they will have a *hard* modal value.

The hard modal values (necessary and impossible) are preserved along of time axis. If, at a given moment, a proposition is necessarily or impossibly true, then it will keep its modal value no matter what would happen in the future, namely, if t_1 comes after t_0, the next relations take place:

$$N_0p \mid\!- N_1p \qquad (12)$$
$$\neg M_0p \mid\!- \neg M_1p$$

Let us demonstrate the first relation:

$N_0p \equiv (T_0 \mid\!- p)$ (13)
$T_1 \equiv (T_0 \,\&\, \dots \,\&\, r(t_1))$
$(N_0p \mid\!- N_1p) \equiv (T_0 \mid\!- p) \mid\!- ((T_0 \,\&\, \dots \,\&\, r(t_1)) \mid\!- p)$
$(N_0p \mid\!- N_1p) \equiv (T_0 \mid\!- p) \mid\!- ((T_0 \mid\!- p) \vee \dots \vee (r(t_1) \mid\!- p))$
The second member of the equivalence is valid according to the logical law $p \mid\!- (p \vee q)$, q.e.d.

Over time, more and more propositions become necessary or impossible, and the class of the possible or contingent propositions diminishes. In this way, time receives an orientation or an arrow, it becomes irreversible. We can anticipate an end of history, when all propositions will be either impossibly or necessarily true and any evolution or change will not be possible. Such a scenario is also theorized by thermodynamics through the *heat death of the universe*. This time, we can talk about the *modal death of the universe*. Both variants are agreeing upon a similar course of the events, foreseeing a future time when changes will not be longer possible and the universe will become frozen.

Let us clarify some relations between the modal and truth values of propositions:

1) If a proposition is necessarily true at a given moment, then it will be true at any future moment:

N_0p, supposition, (14)
$T_0 \mid\!- p$
$T_0(t)$, for any $t > t_0$, since the past cannot be changed,
$(t)p(t)$, *p* is true at any moment after t_0.

In the previous example, after a blue raven occurs, the proposition "Some ravens are not black" becomes true anytime in the future. In this fashion, the forecasting becomes possible. If, at a given moment, a proposition about a future time is necessarily true, then it will be true at that time and we can foresee what will then happen.

2) If a proposition is necessarily true at a given moment, then it is also possibly true[14]:

$$(N_0p \mid\!- M_0p) \equiv ((T_0 \mid\!- p) \mid\!- (T_0 \circ p)) \quad (15)$$

[14] Idem, 4.

If we apply the matriceal decision method[15] on the second member of the equivalence (15) then we obtain the following result:

Table 3. Decision on the formula (15)

	C_0	C_1	C_2	C_3	Conditions
T_0	0	0	1	1	
p	0	1	0	1	
T_0 |- p	1	1	-	1	$C_2 = \varnothing$
	0	0	0	0	$C_2 \neq \varnothing$
T_0 o p	1	1	1	1	$C_3 \neq \varnothing$
	0	0	0	-	$C_3 = \varnothing$
(T_0 |- p) |- (T_0 o p)	0	0	-	-	$C_2 = C_3 = \varnothing$
	1	1	1	1	$C_2 \cup C_3 \neq \varnothing$

We notice that from the necessity of a proposition at the moment t_0 results its possibility only if there will be other contexts after t_0. The consequence relation from necessity to possibility takes place at every moment excepting the moment when the universe will arrive to an end.

A tautology is necessarily true at every moment, but besides tautologies, there are other necessary propositions. In order that a proposition which is necessarily true at a moment be a tautology, it must that the proposition must not had been false before that moment. To prove that relationship, let us decide upon the formula:

N_0p |- (*p* is a tautology) (16)
p is a tautology ≡ (x)(xp)
(T_0 |- p) |- (x)(xp)

Table 4. Decision on the formula (16)

	C_0	C_1	C_2	C_3	Conditions
T_0	0	0	1	1	
p	0	1	0	1	
T_0 |- p	1	1	-	1	$C_2 = \varnothing$
	0	0	0	0	$C_2 \neq \varnothing$
(x)(xp)	-	1	-	1	$C_0 = \varnothing$ & $C_2 = \varnothing$
	0	0	0	0	$C_0 = \varnothing \vee C_2 \neq \varnothing$

[15] J. C. Beall, B. C. v. Fraassen, *Possibilities and Paradox*, 69.

$(T_0 \mid\text{-}\ p) \mid\text{-}\ (x)(xp)$	0	0	0	0	$C_2 = \varnothing \ \&\ C_0 \neq \varnothing$
	1	1	1	1	$C_2 \neq \varnothing \vee C_0 = \varnothing$

If *p* is necessarily true at the moment t_0 and before t_0 the proposition *p* is always true, then *p* is a tautology. All necessary propositions at a given moment are tautologies if, relatively to any previous moment, there were no false propositions. Such a situation is present only at the beginning of the universe, before any context to become real.

At the beginning of time, only the tautologies were necessarily true, namely, since Logic is just the set of all tautologies, it follows that only Logic was necessarily true when the universe begun, and any other proposition has been only contingently true.

3) If a proposition, *p*, is necessarily true at the moment t_0, then it is necessarily true that *p* is necessarily true at t_0, namely, the formula:

$$N_0p \mid\text{-}\ N_0N_0p, \text{ or} \qquad (17)$$
$$(T_0 \mid\text{-}\ p) \mid\text{-}\ (T_0 \mid\text{-}\ (T_0 \mid\text{-}\ p))$$

is valid, according to the next decision table:

Table 5. Decision on the formula (17)

	C_0	**C_1**	**C_2**	**C_3**	**Conditions**
T_0	0	0	1	1	
p	0	1	0	1	
$T_0 \mid\text{-}\ p$	1	1	-	1	$C_2 = \varnothing$
	0	0	0	0	$C_2 \neq \varnothing$
$T_0 \mid\text{-}\ (T_0 \mid\text{-}\ p)$	1	1	-	1	$C_2 = \varnothing$
	1	1	0	0	$C_2 \neq \varnothing$
$(T_0 \mid\text{-}\ p) \mid\text{-}\ (T_0 \mid\text{-}\ (T_0 \mid\text{-}\ p))$	1	1	-	1	$C_2 = \varnothing$
	1	1	1	1	$C_2 \neq \varnothing$

Instead, the formula "$N_0N_0p \mid\text{-}\ N_0p$" is not valid. The past at a certain moment becomes true after that moment. For instance, the proposition "Napoleon was defeated at Waterloo" has become true after the battle and not before it. Therefore, the domain of the past at the moment t_0 is the same with the class of all contexts that are not yet realized at t_0:

$DT_0 = \neg R_0$ (18)

$R_0 = T_0$, hence: $DT_0 = \neg T_0$, or $DT_0 \cup T_0 = X$

III. The End of the World

We have come to the result that the future is contrary to the past, therefore the past cannot happen again. If we have into attention that $IT_0 = (DT_0 \cap \neg R_0)$, then $IT_0 = \neg R_0$, so that, the imaginary domain of T_0, the domain of T_0, and the class of unrealized contexts before t_0 are the same. We can imagine no situation, no evolution of the world, such that the past would not take place or would be modified.

Starting from the expression of necessary true propositions, $N_0p \equiv (T_0 \mid\!- p)$, we will get the result that the imaginary domain of a necessary proposition at a given moment coincides to the class of the unrealized contexts at that moment:

$N_0p \equiv (DT_0 \subset Dp)$ (19)

$N_0p \equiv (\neg R_0 \subset Dp)$, since the domain of antecedent is included in the domain of consequent;

$N_0p \equiv ((\neg R_0 \cap Dp) = \neg R_0)$, but,

$(\neg R_0 \cap Dp) = I_0p$, therefore:

$N_0p \equiv (I_0p = \neg R_0)$, the imaginary domain of a necessary proposition at a given moment coincides with its realizable domain at the same moment.

In other words, a proposition is necessarily true at a moment if and only if there cannot be imagined any future so that the proposition be false or a necessary proposition is true relatively to any imaginable future evolution of the world.

For a possible proposition at the moment t_0 we obtain:

$M_0p \equiv \neg N_0 \neg p$ (20)

$M_0p \equiv \neg((\neg R_0 \cap D\neg p) = \neg R_0)$

$M_0p \equiv ((\neg R_0 \cap D\neg p) \neq \neg R_0)$

$M_0p \equiv (I_0\neg p \neq \neg R_0)$

$\neg R_0 = I_0p \cup I_0\neg p$

$(I_0\neg p \neq \neg R_0) \equiv (I_0p \neq \varnothing)$

$M_0p \equiv (I_0p \neq \varnothing)$

A proposition is possibly true at a moment if its imaginary domain at that moment is not void, respectively, if there is at least an imaginable future where the proposition will be true. Using this result, we can study the relation between the modal and imaginary values of a proposition:

$$M_0p \equiv ((I_0p \neq \varnothing) \ \& \ (I_0\neg p \neq IT_0)) \tag{21}$$
$$N_0p \equiv ((I_0p = IT_0) \ \& \ (I_0\neg p = \varnothing))$$
$$\neg M_0p \equiv ((I_0p = \varnothing) \ \& \ (I_0\neg p = IT_0))$$
$$\neg N_0p \equiv ((I_0p \neq IT_0) \ \& \ (I_0\neg p \neq \varnothing))$$

While for a necessary proposition we could not imagine a future without the truth of that proposition, an impossibly true proposition would be false relatively to any future scenario. At its turn, a contingent proposition will be false for at least some of the imaginary future contexts.

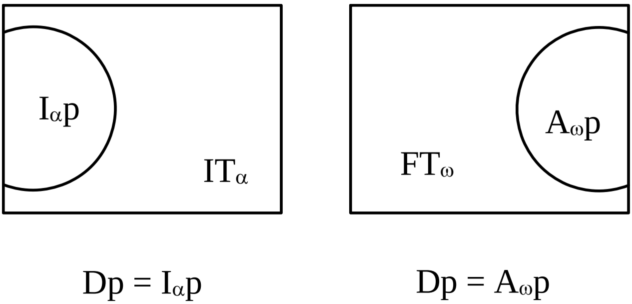

Fig. 2. The beginning and the end of time

At the initial moment of the universe, when all contexts were imaginable, namely, $IT_\alpha = X$, any factual proposition was possible, only the tautologies were necessary and only the contradictions were impossible. Instead, at the end of the universe, when nothing more can be imagined, $IT_\omega = \varnothing$ & $FT_\omega = X$, the possible and the contingency will be vanished and any proposition will be either necessarily or impossibly true, hence no change will be possible, and the evolution of the world will be ended.

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